Uniqueness of solutions for elliptic problems involving the square root of the Laplacian operator

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Abstract

We examine equations of the form

$$(P)_\lambda \qquad \begin{cases} (-\Delta)^{\frac{1}{2}} u = \lambda g(x) f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a parameter and where Ω is a smooth bounded domain in $\mathbb{R}^N$, where $N \geq 2$. Here g is a positive function and f is an increasing, convex function with $f(0) = 1$ and either f blows up at 1 or f is superlinear at ∞. We show that the extremal solution u^* associated with the extremal parameter λ^* is unique. We also show that when f is suitably supercritical and when Ω is star-shaped with respect to the origin that there is a unique solution for small positive λ.

1 Introduction

We are interested in the following nonlocal eigenvalue problem

$$(P)_\lambda \qquad \begin{cases} (-\Delta)^{\frac{1}{2}} u = \lambda g(x) f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $(-\Delta)^{\frac{1}{2}}$ is the square root of the Laplacian operator, $\lambda > 0$ is a parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$ where $N \geq 2$, and where $0 < g(x) \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha$. The nonlinearity f satisfies one of the following two conditions:

(R) f is smooth, increasing and convex on $\mathbb{R}$ with $f(0) = 1$ and f is superlinear at ∞ (i.e. $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$), or

(S) f is smooth, increasing, convex on $[0, 1)$ with $f(0) = 1$ and $\lim_{t \nearrow 1} f(t) = +\infty$.

In this paper we prove there is a unique solution of $(P)_\lambda$ for two parameter ranges: for small λ and for $\lambda = \lambda^*$ where λ^* is the so called extremal parameter associated with $(P)_\lambda$. First, let us to recall various known facts concerning the second order analog of $(P)_\lambda$.

Some notations: $F(t) := \int_0^t f(\tau)d\tau$, $C_f := \int_0^{a_f} f(t)^{-1}dt$ where $a_f = \infty$ (resp. $a_f = 1$) when f satisfies (R) (resp. f satisfies (S)). We say a positive function f defined on an interval I is logarithmically convex (or log convex) provided $u \mapsto \log(f(u))$ is convex on I. Ω will always denote a smooth bounded domain in $\mathbb{R}^N$ where $N \geq 2$.

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1.1 The local eigenvalue problem

For a nonlinearity f which satisfies (R) or (S), the following second order analog of $(P)_\lambda$ with the Dirichlet boundary conditions

$$(Q)_\lambda \qquad \begin{cases} -\Delta u &= \lambda f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

is by now quite well understood whenever Ω is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [3, 4, 5, 14, 15, 16, 18, 20, 21, 2]. We now list the properties one comes to expect when studying $(Q)_\lambda$.

It is well known that there exists a critical parameter $\lambda^* \in (0, \infty)$ such that for all $0 < \lambda < \lambda^*$ there exists a smooth, minimal solution u_λ of $(Q)_\lambda$. Here the minimal solution means in the pointwise sense. In addition for each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing in $(0, \lambda^*)$. This allows one to define the pointwise limit $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ which can be shown to be a weak solution, in a suitably defined sense, of $(Q)_{\lambda^*}$. It is also known that for $\lambda > \lambda^*$ there are no weak solutions of $(Q)_\lambda$. Also, one can show the minimal solution u_λ is a semi-stable solution of $(Q)_\lambda$ in the sense that

$$\int_\Omega \lambda g(x) f'(u_\lambda)\psi^2 \le \int_\Omega |\nabla \psi|^2, \qquad \forall \psi \in H_0^1(\Omega).$$

We now come to the results known for $(Q)_\lambda$ which we are interested in extending to $(P)_\lambda$. In [18] it was shown that the extremal solution u^* is the unique weak solution of $(Q)_{\lambda^*}$. Some of the techniques involve using concave cut offs which do not seem to carry over to the nonlocal setting. Here we use some techniques developed in [1] which were used in studying a fourth order analog of $(Q)_\lambda$. In [11] the uniqueness of the extremal solution for $\Delta^2 u = \lambda e^u$ on radial domains with Dirichlet boundary conditions was shown and this was extended to log convex (see below) nonlinearities in [17]. Some of the methods used in [17] were inspired by the techniques of [1] and so will ours in the case where f satisfies (R). In [8] it was shown that the extremal solution associated with $\Delta^2 u = \lambda(1-u)^{-2}$ on radial domains is unique and our methods for nonlinearities satisfying (S) use some of their techniques.

In [19] and [23] a generalization of $(Q)_\lambda$ was examined. They showed that if f is suitably supercritical near $u = \infty$ and if Ω is a star-shaped domain, then for small $\lambda > 0$ the minimal solution is the unique solution of $(Q)_\lambda$. In [13] this was done for a particular nonlinearity f which satisfies (S). One can weaken the star-shaped assumption and still have uniqueness, see [22], but we do not pursue this approach here. In section 3 we extend these results to $(P)_\lambda$. For more results on uniqueness of solutions for various parameters see [12].

For questions on the regularity of the extremal solution in fourth order problems we direct the interested reader to [10]. We also mention the recent preprint [9] which examines the same issues as this paper but for equations of the form $\Delta^2 u = \lambda f(u)$ in Ω with either the Dirichlet boundary conditions $u = |\nabla u| = 0$ on $\partial\Omega$ or the Navier boundary conditions $u = \Delta u = 0$ on $\partial\Omega$. Elliptic systems of the form $-\Delta u = \lambda f(v)$, $-\Delta v = \gamma g(u)$ in Ω with $u = v = 0$ on $\partial\Omega$ are also examined.

1.2 The nonlocal eigenvalue problem

There is some background material needed related to $(-\Delta)^{\frac{1}{2}}$ if one wishes to examine $(P)_\lambda$. For general questions related to $(-\Delta)^{\frac{1}{2}}$ we refer to [6]. In [7] they examined the problem $(P)_\lambda$ with $(-\Delta)^s$ replacing $(-\Delta)^{\frac{1}{2}}$ and with $g(x) = 1$. They did not investigate the questions we are interested in but they did develop much of the needed theory to examine $(P)_\lambda$ and so we will use many of their results.

There are various ways to make sense of $(-\Delta)^{\frac{1}{2}}u$. Suppose that $u(x)$ is a smooth function defined in Ω which is zero on $\partial\Omega$ and suppose that $u(x) = \sum_k a_k \phi_k(x)$ where (ϕ_k, λ_k) are the eigenpairs of $-\Delta$ in $H_0^1(\Omega)$ which are L^2 normalized. Then one defines

$$(-\Delta)^{\frac{1}{2}} u(x) = \sum_k a_k \sqrt{\lambda_k} \phi_k(x).$$

Another way is to suppose we are given $u(x)$ which is zero on $\partial\Omega$ and we let $u_e = u_e(x,y)$ denote a solution of

$$\begin{cases} \Delta u_e &= 0 \qquad \text{in } \ \mathcal{C} := \Omega \times (0,\infty) \\ u_e &= 0 \qquad \text{on } \ \partial_L \mathcal{C} := \partial\Omega \times (0,\infty) \\ u_e &= u(x) \qquad \text{in } \ \Omega \times \{0\}. \end{cases}$$

Then we define

$$(-\Delta)^{\frac{1}{2}} u(x) = \partial_\nu u_e(x,y)\big|_{y=0},$$

where ν is the outward pointing normal on the bottom of the cylinder, $\mathcal{C}$. We call u_e the harmonic extension of u. We define $H^1_{0,L}(\mathcal{C})$ to be the completion of $C_c^\infty(\Omega \times [0,\infty))$ under the norm $\|u\|^2 := \int_{\mathcal{C}} |\nabla u|^2$. When working on the cylinder generally we will write integrals of the form $\int_{\Omega \times \{y=0\}} \gamma(u_e)$ as $\int_\Omega \gamma(u)$.

Some of our results require one to examine quite weak notions of solutions to $(P)_\lambda$ and so we begin with our definition of a weak solution.

Definition 1. *Given $h(x) \in L^1(\Omega)$ we say that $u \in L^1(\Omega)$ is a weak solution of*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u &= h(x) \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

provided that

$$\int_\Omega u\psi = \int_\Omega h(x)(-\Delta)^{-\frac{1}{2}}\psi \qquad \forall \psi \in C_c^\infty(\Omega).$$

Here $(-\Delta)^{-\frac{1}{2}}\psi$ is given by the function ϕ where

$$\begin{cases} (-\Delta)^{\frac{1}{2}} \phi &= \psi \qquad \text{in } \Omega \\ \phi &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

The following is a weakened special case of a lemma taken from [7].

Lemma 1. *Suppose that $h \in L^1(\Omega)$. Then there exists a unique weak solution u of (1.1). Moreover if $0 \le h$ a.e. then $u \ge 0$ in Ω.*

Definition 2. *Let f be a nonlinearity satisfying (R).*

- *We say that $u(x) \in L^1(\Omega)$ is a weak solution of $(P)_\lambda$ provided $g(x)f(u) \in L^1(\Omega)$, and*

$$\int_\Omega u\psi = \lambda \int_\Omega g(x) f(u) (-\Delta)^{-\frac{1}{2}}\psi \qquad \forall \psi \in C_c^\infty(\Omega).$$

- *We say u is a regular energy solution of $(P)_\lambda$ provided that u is bounded, the harmonic extension u_e of u, is an element of $H^1_{0,L}(\mathcal{C})$ and satisfies*

$$\int_{\mathcal{C}} \nabla u_e \cdot \nabla \phi = \lambda \int_\Omega g(x) f(u) \phi, \tag{1.1}$$

for all $\phi \in H^1_{0,L}(\mathcal{C})$.

- *We say $\overline{u}$ is a regular energy supersolution of $(P)_\lambda$ provided that $0 \le \bar{u}$ is bounded, the harmonic extension of $\bar{u}$ is an element of $H^1_{0,L}(\mathcal{C})$ and satisfies*

$$\int_{\mathcal{C}} \nabla \overline{u}_e \cdot \nabla \phi \ge \lambda \int_\Omega g(x) f(\overline{u}) \phi, \tag{1.2}$$

for all $0 \le \phi \in H^1_{0,L}(\mathcal{C})$.

In the case where f satisfies (S) a few minor changes are needed in the definition of solution. For a weak solution u one requires that $u \leq 1$ a.e. in Ω. For u to be a regular energy solution one requires that $\sup_\Omega u < 1$.

We will need the following monotone iteration result, see [7]. Suppose that $\underline{u}$ and $\overline{u}$ are regular energy sub and supersolutions of $(P)_\lambda$. Then there exists a regular energy solution u of $(P)_\lambda$ and $\underline{u} \leq u \leq \overline{u}$ in Ω. By a regular energy subsolution we are using the natural analog of regular energy supersolution.

We now define the extremal parameter

$$\lambda^* := \sup \left\{ 0 \leq \lambda : \ (P)_\lambda \text{ has a regular energy solution} \right\},$$

and we now show some basic properties.

Lemma 2. *(1) Then $0 < \lambda^*$.*

(2) Then $\lambda^* < \infty$.

(3) For $0 < \lambda < \lambda^*$ there exists a regular energy solution u_λ of $(P)_\lambda$ which is minimal and semi-stable.

(4) For each $x \in \Omega$ the map $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$ and hence the pointwise limit $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is well defined. Then u^* is a weak solution of $(P)_{\lambda^*}$ and satisfies $\int_\Omega g(x) f'(u^*) f(u^*) dx < \infty$.

In this paper we do not need the notion of a semi-stable solution other than for the proof of 4). For the definition of a semi-stable solution one can either use a nonlocal notion, see [7] or instead work on the cylinder which is what we choose to do. We say that a regular energy solution u of $(P)_\lambda$ is semi-stable provided that

$$\int_{\mathcal{C}} |\nabla \phi|^2 \geq \lambda \int_\Omega g(x) f(u) \phi^2 \qquad \forall \phi \in H^1_{0,L}(\mathcal{C}). \tag{1.3}$$

We now prove the lemma.

Proof: (1) Let $\overline{u}$ denote a solution of $(-\Delta)^{\frac{1}{2}} \overline{u} = t g(x)$ with $\overline{u} = 0$ on $\partial\Omega$ where $t > 0$ is small enough such that $\sup_\Omega \overline{u} < 1$. One sees that $\overline{u}$ is a regular energy supersolution of $(P)_\lambda$ provided $t \geq \lambda \sup_\Omega f(\overline{u})$ which clearly holds for small positive λ. Zero is clearly a regular energy subsolution and so we can apply the monotone iteration procedure to obtain a regular energy solution and hence $\lambda^* > 0$.

(2) Suppose that either f satisfies (R) and $C_f < \infty$ or f satisfies (S) and so trivially $C_f < \infty$.

Let u denote a regular energy solution of $(P)_\lambda$ and let u_e denote the harmonic extension. Let ϕ denote the first eigenfunction of $-\Delta$ in $H^1_0(\Omega)$ and let ϕ_e be its harmonic extension; so $\phi_e(x,y) = \phi(x) e^{-\sqrt{\lambda_1} y}$. Multiply $0 = -\Delta u_e$ by $\frac{\phi_e}{f(u_e)}$ and integrate this over the cylinder $\mathcal{C}$ to obtain

$$\int_\Omega \lambda g(x) \phi = \int_{\mathcal{C}} \frac{\nabla u_e \cdot \nabla \phi_e}{f(u_e)} - \int_{\mathcal{C}} \frac{|\nabla u_e|^2 \phi_e f'(u_e)}{f(u_e)^2},$$

and note that the second integral on the right is nonpositive and hence we can rewrite this as

$$\int_\Omega \lambda g(x) \phi \leq \int_{\mathcal{C}} \nabla \phi_e \cdot \nabla h(u_e),$$

where $h(t) = \int_0^t \frac{1}{f(\tau)} d\tau$. Integrating the right hand side by parts we have that it is equal to $\int_\Omega (-\Delta)^{\frac{1}{2}} \phi h(u)$ which is equal to $\sqrt{\lambda_1} \int_\Omega \phi h(u)$. So $h(u) \leq C_f$ and hence we have

$$\lambda \int_\Omega g(x) \phi \leq \sqrt{\lambda_1} C_f \int_\Omega \phi.$$

This shows that $\lambda^* < \infty$. The case where f satisfies (R) and where $C_f = \infty$ needs a separate proof, see the proof of (4). Note that there are examples of f which satisfy (R) and for which $C_f = \infty$, for example $f(t) := (t+1)\log(t+1) + 1$.

(3) The proof in the case where $g(x) = 1$ also works here, see [7].

(4) Again the proof used in the case where $g(x) = 1$ works to show the monotonicity of u_λ, see [7], and hence u^* is well defined. One should note that our notion of a weak solution is more restrictive than what is typically used, ie. we require $g(x)f(u) \in L^1(\Omega)$ where typically one would only require that $\delta(x)g(x)f(u) \in L^1(\Omega)$ where $\delta(x)$ is the distance from x to $\partial\Omega$. Hence here our proof will differ from [7].
Claim: There exists some $C < \infty$ such that

$$\int_\Omega g(x)f'(u_\lambda)f(u_\lambda) \leq C, \tag{1.4}$$

for all $0 < \lambda < \lambda^*$ (at this point we are allowing for the possibility of $\lambda^* = \infty$). We first show that the claim implies that $\lambda^* < \infty$. Note that if $(-\Delta)^{\frac{1}{2}}\phi = g(x)$ with $\phi = 0$ on $\partial\Omega$ then an application of the maximum principle along with the fact that $f(u_\lambda) \geq 1$ gives $u_\lambda \geq \lambda\phi$ in Ω. This along with (1.4) rules out the possibility of $\lambda^* = \infty$. Using a proof similar to the one in [7] one sees that u^* is a weak solution to $(P)_{\lambda^*}$ except for the extra integrability condition $g(x)f(u^*) \in L^1(\Omega)$ that we require. But sending $\lambda \nearrow \lambda^*$ in (1.4) gives us the desired regularity and we are done.
We now prove the claim. Let $u = u_\lambda$ denote the minimal solution of $(P)_\lambda$ and let u_e denote its harmonic extension. Take $\psi := f(u_e) - 1$ in (1.3) (ψ can be shown to be an admissible test function) and write the right hand side as

$$\int_\mathcal{C} \nabla(f(u_e) - 1)f'(u_e) \cdot \nabla u_e,$$

and integrate this by parts. Using $(P)_\lambda$ and after some cancellation one arrives at

$$\int_\mathcal{C} (f(u_e) - 1)f''(u_e)|\nabla u_e|^2 \leq \lambda \int_\Omega g(x)f'(u)f(u). \tag{1.5}$$

Define $H(t) := \int_0^t f''(\tau)(f(\tau) - 1)d\tau$ and so the left hand side of (1.5) can be written as $\int_\mathcal{C} \nabla H(u_e) \cdot \nabla u_e$ and integrating this by parts gives

$$\lambda \int_\Omega g(x)f(u)H(u).$$

Combining this with (1.5) gives

$$\int_\Omega g(x)f(u)H(u) \leq \int_\Omega g(x)f(u)f'(u). \tag{1.6}$$

To complete the proof we show that $H(u)$ dominates $f'(u)$ for big u (resp. u near 1) when f satisfies (R) (resp. (S)). If $0 < T < t$ then one easily sees that

$$H(t) \geq (f(T) - 1)(f'(t) - f'(T)).$$

Using this along with (1.6) and dividing the domain of Ω into regions $\{u \geq T\}$ and $\{u < T\}$ one obtains the claim.

□

2 Uniqueness of the extremal solution

Theorem 1. *Suppose that either f satisfies (R) and is log convex or satisfies (S) and is strictly convex. Then*

(1) There are no weak solutions for $(P)_\lambda$ for any $\lambda > \lambda^*$.

(2) The extremal solution u^* is the unique weak solution of $(P)_{\lambda^*}$.

The following are some properties that the nonlinearity f satisfies.

Proposition 1. *(1) Let f be a log convex nonlinearity which satisfies (R).*

(i) For all $0 < \lambda < 1$ and $\delta > 0$ there exists $k > 0$ such that

$$f(\lambda^{-1}t) + k \geq (1+\delta)f(t) \quad \textit{for all} \;\; 0 \leq t < \infty.$$

(ii) Given $\varepsilon > 0$ there exists $0 < \mu < 1$ such that

$$\mu^2 \left(f(\mu^{-1}t) + \varepsilon \right) \geq f(t) + \frac{\varepsilon}{2} \quad \textit{for all} \;\; 0 \leq t < \infty.$$

(iii) Then f is strictly convex.

(2) Let f be a nonlinearity which satisfies (S).

(i) Given $\varepsilon > 0$ there exists $0 < \mu < 1$ such that

$$\mu \left(f(\mu^{-1}t) + \varepsilon \right) \geq f(t) + \frac{\varepsilon}{2} \quad \textit{for all} \;\; 0 \leq t \leq \mu.$$

(ii) Then $\lim_{t \nearrow 1} \frac{f(t)}{F(t)} = \infty$ where $F(t) := \int_0^t f(\tau) d\tau$.

Proof. See [1], [17] for the proof of (1)-(i) and (1)-(ii). Part (1)-(iii) is trivial.
(2)-(i) Set $h(t) := \mu\{f(\mu^{-1}t) + \varepsilon\} - f(t) - \frac{\varepsilon}{2}$ and note that $h'(t) \geq 0$ for all $0 \leq t \leq \mu$ and that $h(0) > 0$ for μ sufficiently close to 1 which gives us the desired result.
(2)-(ii) Let $0 < t < 1$ and we use a Riemann sum with right hand endpoints to approximate $F(t)$. So for any positive integer n we have

$$F(t) \;\; \leq \;\; \frac{t}{n} \sum_{k=1}^{n} f(\frac{kt}{n}) \leq \frac{t(n-1)}{n} f(\frac{(n-1)t}{n}) + \frac{t}{n} f(t),$$

and so

$$\limsup_{t \nearrow 1} \frac{F(t)}{f(t)} \leq \frac{1}{n},$$

but since n is arbitrary we have the desired result.

□

The following is an essential step in proving Theorem 1. We give the proof of this lemma later.

Lemma 3. *Suppose that f is log convex and satisfies (R) or f satisfies (S). Suppose $\varepsilon > 0$ and that $0 \leq \tau$ is a weak solution of*

$$\begin{cases} (-\Delta)^{\frac{1}{2}} \tau = l(x) & \textit{in } \Omega \\ \tau = 0 & \textit{on } \partial\Omega, \end{cases}$$

where $g(x)\left(f(\tau) + \varepsilon\right) \leq l(x) \in L^1(\Omega)$. Then there exists a regular energy solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = g(x)\left(f(u) + \frac{\varepsilon}{2}\right) & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Proof of Theorem 1: Without loss of generality assume that $\lambda^*=1$ and let u^* denote the extremal solution of $(P)_{\lambda^*}$. Suppose that v is also a weak solution of $(P)_{\lambda^*}$ and v is not equal to u^*. Set $\Omega_0 := \{x \in \Omega : u^*(x) \neq v(x),\ u^*(x), v(x) \in \mathbb{R}\}$ (resp. $\Omega_0 = \{x \in \Omega : u^*(x) \neq v(x),\ u^*(x), v(x) < 1\}$) when f satisfies (R) (resp. (S)) and note that $|\Omega_0| > 0$. Define

$$h(x) := \begin{cases} \frac{f(u^*(x))+f(v(x))}{2} - f(\frac{u^*(x)+v(x)}{2}) & x \in \Omega_0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that by the strict convexity of f, which we obtain either by hypothesis or by Proposition 1 1 (iii), we have $0 \leq h$ in Ω and $h > 0$ in Ω_0. Also note that $h \in L^1(\Omega)$. Define $z := \frac{u^*+v}{2}$. Since u^* and v are weak solutions of $(P)_{\lambda^*}$, z is a weak solution of

$$(-\Delta)^{\frac{1}{2}} z = g(x)f(z) + g(x)h(x) \qquad \text{in } \Omega,$$

with $z=0$ on $\partial\Omega$. From now on we omit the boundary values since they will always be zero unless otherwise mentioned. Let χ and ϕ denote weak solutions of $(-\Delta)^{\frac{1}{2}}\chi = g(x)h(x)$ and $(-\Delta)^{\frac{1}{2}}\phi = g(x)$ in Ω. By taking $\varepsilon > 0$ small enough one has that $\chi \geq \varepsilon\phi$ in Ω. Set $\tau := z + \varepsilon\phi - \chi$ and note that τ is a weak solution of

$$(-\Delta)^{\frac{1}{2}}\tau = g(x)(f(z)+\varepsilon) \geq 0 \qquad \text{in } \Omega,$$

and by Lemma 1 we have that $0 \leq \tau$. Moreover, from the fact that $\tau \leq z$ in Ω we have

$$g(x)(f(\tau)+\varepsilon) \leq (-\Delta)^{\frac{1}{2}}\tau \in L^1(\Omega).$$

Applying Lemma 3, there exists a regular energy solution u of

$$(-\Delta)^{\frac{1}{2}} u = g(x)(f(u) + \frac{\varepsilon}{2}) \qquad \text{in } \Omega.$$

Set $w := u + \alpha u - \frac{\varepsilon}{2}\phi$ where $\alpha > 0$ is chosen small enough such that $\alpha u \leq \frac{\varepsilon}{2}\phi$ in Ω. A straightforward computation shows that w is a regular energy solution of

$$(-\Delta)^{\frac{1}{2}} w = (1+\alpha)g(x)f(u) + \frac{\varepsilon}{2}\alpha g(x) \qquad \text{in } \Omega,$$

and that $w \leq u$ in Ω. By Lemma 1 we also have $0 \leq w$ in Ω. From this we see that w is a regular energy supersolution of

$$(-\Delta)^{\frac{1}{2}} w \geq (1+\alpha)g(x)f(w) \qquad \text{in } \Omega,$$

with zero boundary conditions. We now apply the monotone iteration argument to obtain a regular energy solution $\tilde{u}$ of $(-\Delta)^{\frac{1}{2}}\tilde{u} = (1+\alpha)g(x)f(\tilde{u})$ in Ω which contradicts the fact that $\lambda^* = 1$. So, we have shown that $|\Omega_0| = 0$ and so $u^* = v$ a.e. in Ω.

□

Proof of Lemma 3: Let $\varepsilon > 0$ and suppose that $0 \leq \tau \in L^1(\Omega)$ is a weak solution of $(-\Delta)^{\frac{1}{2}}\tau = l(x)$ in Ω where $0 \leq g(x)(f(\tau)+\varepsilon) \leq l(x)$ in Ω. As in the proof of Theorem 1, we omit the boundary values since they will always be Dirichlet boundary conditions and we also assume that $\lambda^* = 1$. First, assume that f is a log convex nonlinearity which satisfies (R). Let $u_0 := \tau$ and let u_1, u_2, u_3 be weak solutions of

$$(-\Delta)^{\frac{1}{2}} u_1 = \mu g(x)(f(u_0)+\varepsilon) \qquad \text{in } \Omega,$$
$$(-\Delta)^{\frac{1}{2}} u_2 = \mu g(x)(f(u_1)+\varepsilon) \qquad \text{in } \Omega,$$
$$(-\Delta)^{\frac{1}{2}} u_3 = \mu g(x)(f(u_2)+\varepsilon) \qquad \text{in } \Omega,$$

where $0 < \mu < 1$ is the constant given in Proposition 1 such that $\mu^2\left(f(\frac{t}{\mu})+\varepsilon\right) \geq f(t) + \frac{\varepsilon}{2}$ for all $t \geq 0$. One easily sees that $u_2 \leq u_1 \leq \mu u_0$. Now note that

$$\begin{aligned} (-\Delta)^{\frac{1}{2}} u_1 &= \mu g(x)(f(u_0)+\varepsilon) \\ &\geq \mu g(x)\left(f(\frac{u_1}{\mu})+\varepsilon\right). \end{aligned} \tag{2.1}$$

By Proposition 1 with $\delta := 2N-1 > 0$ and $0 < \lambda = \mu < 1$ there exists some $k > 0$ such that

$$f(\frac{u_1}{\mu}) \geq 2Nf(u_1) - k,$$

hence one can rewrite (2.1) as

$$(-\Delta)^{\frac{1}{2}} u_1 \quad \geq \quad \mu g(x) \left(2Nf(u_1) - k + \varepsilon\right).$$

We let ϕ be as in the proof of Theorem 1 and examine $u_1 + t\phi$ where $t > 0$ is to be picked later. Note that

$$\begin{aligned}(-\Delta)^{\frac{1}{2}}(u_1 + t\phi) &= (-\Delta)^{\frac{1}{2}} u_1 + tg(x) \\ &\geq 2N\mu g(x) \left(f(u_1) + \varepsilon\right) + mg(x),\end{aligned}$$

where $m := t - \mu k + \varepsilon\mu(1-2N)$ and we now pick $t > 0$ big enough such that $m = 0$. Therefore, from the definition of u_2 we have that

$$(-\Delta)^{\frac{1}{2}}(u_1 + t\phi) \geq 2N\ (-\Delta)^{\frac{1}{2}} u_2 \qquad \text{in } \Omega.$$

So, from the maximum principle we get

$$u_2 \leq \frac{1}{2N}(u_1 + t\phi) \qquad \text{in } \Omega.$$

Since f is log convex, there is some smooth, convex increasing function β with $\beta(0) = 0$ and $f(t) = e^{\beta(t)}$. By the convexity of β and since $\beta(0) = 0$, we have

$$\beta(u_2) \leq \frac{1}{2N}\beta(u_1 + t\phi) \leq \frac{1}{2N}\beta(\mu u_0 + t\phi),$$

but

$$\beta(\mu u_0 + t\phi) = \beta(\mu u_0 + (1-\mu)\frac{t\phi}{1-\mu}) \leq \mu\beta(u_0) + (1-\mu)\beta(\frac{t\phi}{1-\mu}).$$

From this we can conclude

$$f(u_2)^{2N} \leq e^{\mu\beta(u_0)} e^{(1-\mu)\beta(\frac{t\phi}{1-\mu})} \leq f(u_0) f(\frac{t\phi}{1-\mu})^{1-\mu}.$$

So, we see that $g(x)f(u_2)^{2N} \leq Cg(x)f(u_0) \in L^1(\Omega)$ for some large constant C.

Since $g(x)$ is bounded, we conclude that $g(x)f(u_2) \in L^{2N}(\Omega)$. But u_3 satisfies $(-\Delta)^{\frac{1}{2}} u_3 = \mu g(x)(f(u_2) + \varepsilon)$ in Ω and so by elliptic regularity we have that u_3 is bounded (since the right hand side is an element of $L^p(\Omega)$ for some $p > N$) and now we use the fact that $0 \leq u_3 \leq u_2$ and the monotone iteration argument to obtain a regular energy solution w to $(-\Delta)^{\frac{1}{2}} w = \mu g(x)(f(w) + \varepsilon)$ in Ω.

Now, set $\xi := \mu w$ and note that ξ is a regular energy solution of

$$(-\Delta)^{\frac{1}{2}} \xi = \mu^2 g(x) \left(f(\frac{\xi}{\mu}) + \varepsilon\right) \qquad \text{in } \Omega,$$

and from Proposition 1, we have

$$(-\Delta)^{\frac{1}{2}} \xi \geq g(x) \left(f(\xi) + \frac{\varepsilon}{2}\right) \qquad \text{in } \Omega,$$

and so by an iteration argument, we have the desired result.

Now, assume that f satisfies (S). In this case, the proof is much simpler. Define $w := \mu\tau$ where $0 < \mu < 1$ is from Proposition 1. Then note that $0 \leq w \leq \mu$ a.e. and

$$
\begin{aligned}
(-\Delta)^{\frac{1}{2}} w = \mu l(x) &\geq \mu g(x)(f(\frac{w}{\mu}) + \varepsilon) \\
&\geq g(x)(f(w) + \frac{\varepsilon}{2}).
\end{aligned}
$$

Hence, w is a regular energy supersolution of

$$
(-\Delta)^{\frac{1}{2}} w \geq g(x)(f(w) + \frac{\varepsilon}{2}),
$$

and we have the desired result after an application of the monotone iteration argument.

□

3 Uniqueness of solutions for small λ

In this section we prove uniqueness theorems for equation $(P)_\lambda$ for small enough λ. Throughout this section we assume that $g = 0$ on $\partial\Omega$. We need the following regularity result.

Proposition 2. *[6] Let $\alpha \in (0,1)$, Ω be a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^N$ and suppose that u is a weak solution of $(-\Delta)^{\frac{1}{2}} u = h(x)$ in Ω with $u = 0$ on $\partial\Omega$. Then*

(1) Suppose that $h \in L^\infty(\Omega)$. Then $u_e \in C^{0,\alpha}(\overline{\mathcal{C}})$ hence $u \in C^{0,\alpha}(\overline{\Omega})$.

(2) Suppose that $h \in C^{k,\alpha}(\overline{\Omega})$ where $k = 0$ or $k = 1$ and $h = 0$ on $\partial\Omega$. Then $u_e \in C^{k+1,\alpha}(\overline{\mathcal{C}})$ hence $u \in C^{k+1,\alpha}(\overline{\Omega})$.

Using this one easily obtains the following:

Corollary 1. *For each $0 < \lambda < \lambda^*$ the minimal solution of $(P)_\lambda$, u_λ, belongs to $C^{2,\alpha}(\overline{\Omega})$. In addition $u_\lambda \to 0$ in $C^1(\overline{\Omega})$ as $\lambda \to 0$.*

We now come to our main theorem of this section.

Theorem 2. *Suppose that Ω is a star-shaped domain with respect to the origin and set $\gamma := \sup_\Omega \frac{x \cdot \nabla g(x)}{g(x)}$.*

(1) Suppose that f satisfies (R) and that

$$
\limsup_{t \to \infty} \frac{F(t)}{f(t)t} < \frac{N-1}{2(N+\gamma)}. \tag{3.1}
$$

Then for sufficiently small λ, u_λ is the unique regular energy solution of $(P)_\lambda$.

(2) Suppose that f satisfies (S). Then for sufficiently small λ, u_λ is the unique regular energy solution $(P)_\lambda$.

Proof: Let f satisfy (R) and (3.1) or let f satisfy (S) and suppose that u is a second regular energy solution of $(P)_\lambda$ which is different from the minimal solution u_λ. Set $v := u - u_\lambda$ and note that $v \geq 0$ by the minimality of u_λ and $v \neq 0$ since u is different from the minimal solution.

A computation shows that v satisfies the equation

$$
(-\Delta)^{\frac{1}{2}} v = \lambda g(x) \left\{ f(u_\lambda + v) - f(u_\lambda) \right\}. \tag{3.2}
$$

Applying Proposition 2 to u and u_λ separately shows that $v_e \in C^{2,\alpha}(\overline{\mathcal{C}})$.

A computation shows the following identity holds

$$
\operatorname{div}\{(z, \nabla v_e)\nabla v_e - z\frac{|\nabla v_e|^2}{2}\} + \frac{N-1}{2}|\nabla v_e|^2 = (z, \nabla v_e)\Delta v_e,
$$

where $z=(x,y)$. Integrating this identity over $\Omega\times(0,R)$ we end up with

$$\frac{1}{2}\int_{\partial\Omega\times(0,R)}|\nabla v_e|^2\ x\cdot\nu+\int_{\Omega}x\cdot\nabla_x v_e\ \partial_\nu v_e+\frac{N-1}{2}\int_{\Omega\times(0,R)}|\nabla v_e|^2+\varepsilon(R)=0, \tag{3.3}$$

where

$$\varepsilon(R):=\int_{\Omega\times\{y=R\}}(x\cdot\nabla_x v_e+R\ \partial_y v_e)\,\partial_y v_e-\frac{R}{2}|\nabla v_e|^2.$$

One can show that $\varepsilon(R)\to 0$ as $R\to\infty$, for details on this and the above calculations see [24]. Sending $R\to\infty$ and since Ω is star-shaped with respect to the origin, we have

$$\frac{N-1}{2}\int_{\mathcal{C}}|\nabla v_e|^2\le-\int_{\Omega}x\cdot\nabla_x v\ \partial_\nu v_e,$$

and after using (3.2) one obtains

$$\frac{N-1}{2}\int_{\mathcal{C}}|\nabla v_e|^2\le-\lambda\int_{\Omega}x\cdot\nabla_x v\ g(x)\{f(u_\lambda+v)-f(u_\lambda)\}. \tag{3.4}$$

We now compute the right hand side of (3.4). Set $h(x,\tau):=f(u_\lambda(x)+\tau)-f(u_\lambda(x))$ and let $H(x,t)=\int_0^t h(x,\tau)d\tau$. For this portion of the proof we are working on Ω and hence all gradients are with respect to the x variable. To clarify our notation note that the chain rule can be written as

$$\nabla H(x,v)=\nabla_x H(x,v)+h(x,v)\nabla v,$$

where we recall $v=v(x)$. Some computations now show that

$$H(x,t)=F(u_\lambda+t)-F(u_\lambda)-f(u_\lambda)t,$$

and

$$\nabla_x H(x,t)=\{f(u_\lambda+t)-f(u_\lambda)-f'(u_\lambda)t\}\nabla u_\lambda,$$

and so the right hand side of (3.4) can be written as

$$\begin{aligned}-\lambda\int_{\Omega}g(x)\{f(u_\lambda+v)-f(u_\lambda)\}x\cdot\nabla v &= -\lambda\int_{\Omega}g(x)h(x,v)x\cdot\nabla v\\ &= -\lambda\int_{\Omega}g(x)x\cdot\{\nabla H(x,v)-\nabla_x H(x,v)\}\\ &= \lambda\int_{\Omega}g(x)x\cdot\nabla_x H(x,v)+\lambda N\int H(x,v)g(x)\\ &\quad +\lambda\int_{\Omega}H(x,v)x\cdot\nabla g(x).\end{aligned}$$

Therefore, (3.4) can be written as

$$\begin{aligned}\frac{N-1}{2}\int_{\mathcal{C}}|\nabla v_e|^2 &\le \lambda\int_{\Omega}x\cdot\nabla u_\lambda g(x)\{f(u_\lambda+v)-f(u_\lambda)-f'(u_\lambda)v\}\\ &\quad +N\lambda\int_{\Omega}g(x)\{F(u_\lambda+v)-F(u_\lambda)-f(u_\lambda)v\}\\ &\quad +\lambda\int_{\Omega}x\cdot\nabla g(x)\{F(u_\lambda+v)-F(u_\lambda)-f(u_\lambda)v\}.\end{aligned} \tag{3.5}$$

We now assume we are in case (1). Let α be such that

$$\limsup_{\tau\to\infty}\frac{F(\tau)}{\tau f(\tau)}<\alpha<\frac{N-1}{2(N+\gamma)},$$

so there exists some $\tau_0 > 0$ such that $F(\tau) < \alpha\tau f(\tau)$ for all $\tau \geq \tau_0$. Let $0 < \theta < 1$ be such that $\frac{\theta(N-1)}{2} - \alpha(N+\gamma) > 0$ and we now decompose the left hand side of (3.5) into the convex combination

$$\frac{\theta(N-1)}{2}\int_{\mathcal{C}}|\nabla v_e|^2 + \frac{(N-1)(1-\theta)}{2}\int_{\mathcal{C}}|\nabla v_e|^2. \tag{3.6}$$

Using the following trace theorem: there exists some $\tilde{C} > 0$ such that

$$\int_{\mathcal{C}}|\nabla w|^2 \geq \tilde{C}\int_{\Omega} w^2, \qquad \forall w \in H^1_{0,L}(\mathcal{C}), \tag{3.7}$$

one sees that (3.6) is bounded below by

$$\frac{\theta(N-1)}{2}\int_{\mathcal{C}}|\nabla v_e|^2 + C\int_{\Omega} v^2.$$

By taking $C > 0$ smaller if necessary one can bound this from below by

$$\frac{\theta(N-1)}{2}\int_{\mathcal{C}}|\nabla v_e|^2 + C\int_{\Omega} g(x)v^2,$$

and after using (3.2), this last quantity is equal to

$$\frac{\lambda\theta(N-1)}{2}\int_{\Omega} g(x)\{f(u_\lambda+v) - f(u_\lambda)\}v + C\int_{\Omega} g(x)v^2. \tag{3.8}$$

Substituting (3.8) into (3.4) and rearranging one arrives at an inequality of the form

$$\int_{\Omega} g(x)T_\lambda(x,v) \leq 0,$$

where

$$\begin{aligned} T_\lambda(x,\tau) \quad = \quad & \frac{\theta(N-1)}{2}\{f(u_\lambda+\tau) - f(u_\lambda)\}\tau + \frac{C}{\lambda}\tau^2 \\ & -N\{F(u_\lambda+\tau) - F(u_\lambda) - f(u_\lambda)\tau\} \\ & -\frac{x\cdot\nabla g}{g}\{F(u_\lambda+\tau) - F(u_\lambda) - f(u_\lambda)\tau\} \\ & -x\cdot\nabla u_\lambda\{f(u_\lambda+\tau) - f(u_\lambda) - f'(u_\lambda)\tau\}. \end{aligned}$$

To obtain a contradiction we show that for sufficiently small $\lambda > 0$ that $T_\lambda(x,\tau) > 0$ on $(x,\tau) \in \Omega \times (0,\infty)$ and hence we must have that $v = 0$. Define

$$\begin{aligned} S_\lambda(x,\tau) \quad = \quad & \frac{\theta(N-1)}{2}\{f(u_\lambda+\tau) - f(u_\lambda)\}\tau + \frac{C}{\lambda}\tau^2 \\ & -(N+\gamma)\{F(u_\lambda+\tau) - F(u_\lambda) - f(u_\lambda)\tau\} \\ & -\varepsilon_\lambda\{f(u_\lambda+\tau) - f(u_\lambda) - f'(u_\lambda)\tau\}. \end{aligned}$$

where $\varepsilon_\lambda := \|\nabla u_\lambda \cdot x\|_{L^\infty}$. Note that since f is increasing and convex that $T_\lambda(x,\tau) \geq S_\lambda(x,\tau)$ for all $\tau \geq 0$. We now show the desired positivity for S_λ and to do this we examine large and small τ separately.

Large τ: Take $\tau \geq \tau_0$ and $0 < \lambda \leq \frac{\lambda^*}{2}$. Since f is convex and increasing

$$\begin{aligned} S_\lambda(x,\tau) \quad \geq \quad & \frac{\theta(N-1)}{2}f(u_\lambda+\tau)\tau - (N+\gamma)F(u_\lambda+\tau) \\ & -\varepsilon_\lambda f(u_\lambda+\tau) + \frac{C}{\lambda}\tau^2 \\ & -\frac{\theta(N-1)}{2}f(u_\lambda)\tau, \end{aligned} \tag{3.9}$$

but $F(u_\lambda+\tau)<\alpha(u_\lambda+\tau)f(u_\lambda+\tau)$ for all $\tau\ge\tau_0$ and so the right hand side of (3.9) is bounded below by

$$
\begin{aligned}
& f(u_\lambda+\tau)\left[\tau\left\{\frac{\theta(N-1)}{2}-(N+\gamma)\alpha\right\}-\varepsilon_\lambda-(N+\gamma)\alpha u_\lambda\right] \\
& -\frac{\theta(N-1)}{2}f(u_\lambda)\tau+\frac{C}{\lambda}\tau^2.
\end{aligned}
$$

Using the fact that f is superlinear at ∞ there exists some $\tau_1\ge\tau_0$ such that $S_\lambda(x,\tau)>0$ for all $\tau\ge\tau_1$ and $0<\lambda\le\frac{\lambda^*}{2}$.
Small τ: Let $0<\lambda_0<\frac{\lambda^*}{2}$ be such that $\|u_\lambda\|_{L^\infty}\le 1$. Using the convexity and monotonicity of f and Taylor's Theorem there exists some $C_1>0$ such that

$$
F(u_\lambda+\tau)-F(u_\lambda)-f(u_\lambda)\tau\le C_1\tau^2, \qquad f(u_\lambda+\tau)-f(u_\lambda)-f'(u_\lambda)\tau\le C_1\tau^2,
$$

for all $0\le\tau\le\tau_0$, $0<\lambda\le\lambda_0$ and $x\in\Omega$. Noting that the first term of $S_\lambda(x,\tau)$ is positive for $\tau>0$ one sees that for all $0<\tau\le\tau_0$, $x\in\Omega$ and $0<\lambda<\lambda_0$ one has the lower bound

$$
S_\lambda(x,\tau)\ge\frac{C}{\lambda}\tau^2-(N+\gamma+\varepsilon_\lambda)C_1\tau^2,
$$

and hence by taking λ smaller if necessary we have the desired result.
(2) We now assume that f satisfies (S). One uses a similar approach to arrive at an inequality of the form

$$
\int_\Omega T_\lambda(x,v)\le 0,
$$

where as before $v=u-u_\lambda\ge 0$ and where we assume that $v\ne 0$. To arrive at a contradiction we show that for sufficiently small λ that $T_\lambda(x,\tau)>0$ for all $x\in\Omega$ and for all $0<\tau<1-u_\lambda(x)$. Again the idea is to break the interval into 2 regions. For τ such that $\tau+u_\lambda(x)$ close to 1 we use Proposition 1, 2 (ii) to see the desired positivity. For the remainder of the interval we again use Taylor's Theorem.

□

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